

HOMOMORPHISMS AND PRINCIPAL CONGRUENCES OF BOUNDED LATTICES

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ABSTRACT. Two years ago, I characterized the order $\text{Princ } L$ of principal congruences of a bounded lattice L as a bounded order.

If K and L are bounded lattices and φ is a $\{0, 1\}$ -homomorphism of K into L , then there is a natural isotone $\{0, 1\}$ -map φ_{Hom} from $\text{Princ } K$ into $\text{Princ } L$.

We prove the converse: For bounded orders P and Q and an isotone $\{0, 1\}$ -map ψ of P into Q , we represent P and Q as $\text{Princ } K$ and $\text{Princ } L$ for bounded lattices K and L with a $\{0, 1\}$ -homomorphism φ of K into L , so that ψ is represented as φ_{Hom} .

1. INTRODUCTION

In my paper [6], I prove the Characterization Theorem of the Order of Principal Congruences:

Theorem 1. *Let L be a bounded lattice and let $\text{Princ } L$ denote the order of principal congruences of L . The order $\text{Princ } L$ can be characterized as a bounded order.*

G. Czédli [2] and [3] extended this result to a bounded lattice and a $\{0, 1\}$ -sublattice. Let K be a $\{0, 1\}$ -sublattice of a bounded lattice L . Then the map

$$(1) \quad \psi_{\text{Sub}}: \text{con}_K(x, y) \mapsto \text{con}_L(x, y)$$

is an isotone $\{0, 1\}$ -map of $\text{Princ } K$ into $\text{Princ } L$. Observe that the $\{0, 1\}$ -map ψ_{Sub} is 0-separating, that is, $\mathbf{0}_K$ is the only element mapped by ψ_{Sub} to $\mathbf{0}_L$.

Now we can state G. Czédli's result.

Theorem 2. *Let P and Q be bounded orders and let ψ be an isotone 0-separating $\{0, 1\}$ -map from P into Q . Then there exist a bounded lattice L , a $\{0, 1\}$ -sublattice K of L , so that P , Q , and ψ are represented by $\text{Princ } K$, $\text{Princ } L$, and ψ_{Sub} up to isomorphism.*

Theorem 1 follows from Theorem 2 with $P = Q$ and ψ the identity map.

In this note we take up the analogous problem with homomorphic images rather than sublattices. We start with the following observation.

Lemma 3. *Let K and L be bounded lattices and let φ be a $\{0, 1\}$ -homomorphism of K into L . Define*

$$(2) \quad \psi_{\text{Hom}}: \text{con}_K(a, b) \mapsto \text{con}_L(\varphi(a), \varphi(b))$$

for $a \leq b \in K$. Then φ_{Hom} is an isotone $\{0, 1\}$ -map of $\text{Princ } K$ into $\text{Princ } L$.

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Now we state our main result, the Representation Theorem for Order Triples.

Theorem 4. *Let P and Q be bounded orders and let ψ be an isotone $\{0, 1\}$ -map from P into Q . Then there exist bounded lattices K , L , and a $\{0, 1\}$ -homomorphism φ of K into L , so that P , Q , and ψ are represented by $\text{Princ } K$, $\text{Princ } L$, and ψ_{Hom} , up to isomorphism.*

We will consider *lattice-triples*: $\mathcal{L} = (K, L, \varphi)$, where K and L are bounded lattices and φ is a $\{0, 1\}$ -homomorphism of K into L . Similarly, we consider *order-triples* $\mathcal{P} = (P, Q, \psi)$, where P and Q are bounded orders and ψ is an isotone $\{0, 1\}$ -map of P into Q . By Lemma 3, a lattice-triple \mathcal{L} defines an order-triple \mathcal{P} in the natural way: $P = \text{Princ } K$, $Q = \text{Princ } L$, and $\psi = \psi_{\text{Hom}}$; we shall use the notation $\mathcal{P}^3(\mathcal{L})$ for this order-triple. A *representable* order-triple \mathcal{P} arises from a lattice-triple \mathcal{L} as $\mathcal{P}^3(\mathcal{L})$.

Now we restate the Representation Theorem.

Theorem 5. *Every order-triple is representable.*

The proof of this theorem relies on the construction in [6] to prove Theorem 1. To keep this paper short, we assume familiarity this construction. We also assume familiarity the basic concepts and notation of this field, see any one of my books [5]–[7].

In Section 2, we verify some elementary facts, including that the map in (2) is well-defined. Section 3 describes the main step in the proof of the Representation Theorem, proving it in a very special case. Section 4 combines the result in Section 3 with Czédli's Theorem 2 to verify the Representation Theorem.

We list some open problems in Section 5. In Appendix A, we point out that a lattice construction of Czédli's can be made smaller.

2. PRINCIPAL CONGRUENCES AND HOMOMORPHISMS

We start by restating two well-known results, see for instance, Lemma 229 and Theorem 230 in [5].

Lemma 6. *Let L be a lattice, $a, b, c, d \in L$ with $a \leq b$ and $c \leq d$. Then $[a, b]$ is congruence-projective to $[c, d]$ iff there is an integer m and there are elements $p_0, \dots, p_{m-1} \in L$ such that*

$$(3) \quad t(a, p_0, \dots, p_{m-1}) = c,$$

$$(4) \quad t(b, p_0, \dots, p_{m-1}) = d,$$

where t is defined by

$$t(x, y_0, \dots, y_{m-1}) = \dots(((x \vee y_0) \wedge y_1) \vee y_2) \wedge \dots.$$

Lemma 7. *Let L be a lattice and let $a \leq b$ and $c \leq d$ in L . Then*

$$(5) \quad c \equiv d \pmod{\text{con}(a, b)}$$

iff, for some ascending sequence

$$(6) \quad c = e_0 \leq e_1 \leq \dots \leq e_n = d,$$

the congruence-projectivities

$$(7) \quad [a, b] \Rightarrow [e_j, e_{j+1}]$$

hold for all $j = 0, \dots, n-1$.

The next three lemmas are easy to prove.

Lemma 8. *Let K and L be lattices and let φ be a homomorphism of K into L . If $a \leq b$, $x \leq y$, and $[a, b] \Rightarrow [x, y]$ in K , then*

$$(8) \quad [\varphi(a), \varphi(b)] \Rightarrow [\varphi(x), \varphi(y)]$$

holds in L .

Proof. By Lemma 6, if $[a, b] \Rightarrow [x, y]$, then there is an integer m and there are elements $p_0, \dots, p_{m-1} \in K$ such that (3) and (4) hold. Since φ is a homomorphism, we get that

$$(9) \quad t(\varphi(a), \varphi(p_0), \dots, \varphi(p_{m-1})) = \varphi(x),$$

$$(10) \quad t(\varphi(b), \varphi(p_0), \dots, \varphi(p_{m-1})) = \varphi(y).$$

Again, by Lemma 7, (9) and (10) imply that (8) holds. \square

Lemma 9. *Let K and L be lattices and let φ be a homomorphism of K into L . If $a \leq b$, $x \leq y$, and $x \equiv y \pmod{\text{con}(a, b)}$ in K , then*

$$(11) \quad \varphi(x) \equiv \varphi(y) \pmod{\text{con}(\varphi(a), \varphi(b))}$$

holds in L .

Proof. By Lemma 7, there is a sequence $c = e_0 \leq e_1 \leq \dots \leq e_n = d$, such that (7) holds. By Lemma 8,

$$(12) \quad [\varphi(a), \varphi(b)] \Rightarrow [\varphi(e_j), \varphi(e_{j+1})]$$

for all $j = 0, \dots, n-1$. By Lemma 7, (11) holds. \square

We rewrite Lemma 9 as follows.

Let K and L be lattices, let φ be a homomorphism of K into L , and let us assume that $a, b, c, d \in K$. Then

$$(13) \quad \text{con}(a, b) \geq \text{con}(c, d) \text{ in } K \text{ implies that } \text{con}(\varphi(a), \varphi(b)) \geq \text{con}(\varphi(c), \varphi(d)) \text{ in } L.$$

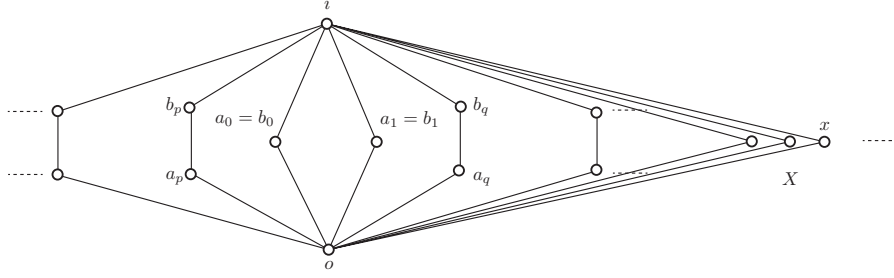
Let us call the lattice-triple $\mathcal{L} = (K, L, \varphi)$ *surjective*, if φ maps K onto L . Similarly, an order-triple $\mathcal{P} = (P, Q, \psi)$ is *surjective*, if ψ maps P onto Q . An order-triple $\mathcal{P} = (P, Q, \psi)$ has a *surjective representation* if there is a surjective lattice-triple representing it.

Lemma 10. *If the order-triple $\mathcal{P} = (P, Q, \psi)$ has a surjective representation $\mathcal{L} = (K, L, \varphi)$, then \mathcal{P} is surjective.*

Proof. Let $u, v \in L$. Since φ is surjective, there are elements $a, b \in K$ with $\varphi(a) = u$ and $\varphi(b) = v$. It follows from (2) that $\varphi_{\text{Hom}}(\text{con}(a, b)) = \text{con}(u, v)$, so the map $\varphi_{\text{Hom}} = \psi$ is surjective. \square

Now we prove Lemma 3. Since K and L are bounded lattices and φ is a $\{0, 1\}$ -homomorphism of K into L , it follows that φ_{Hom} is a $\{0, 1\}$ -map.

By Lemma 10, the map φ_{Hom} is surjective. Applying (13) twice to $\text{con}(a, b) = \text{con}(c, d)$, we conclude that $\text{con}(\varphi(a), \varphi(b)) = \text{con}(\varphi(c), \varphi(d))$ in L , proving that φ_{Hom} is a map. (13) also verifies that φ_{Hom} is isotone, concluding the proof of Lemma 3.

FIGURE 1. The lattice $\text{Frame}_X P$

3. THE MAIN STEP

The main step in the proof of the Representation Theorem is its verification in a very special case.

We need some notation. For an order-triple $\mathcal{P} = (P, Q, \psi)$, we define

$$\text{Top } \mathcal{P} = \{x \in P \mid \psi(x) > 0_Q\} \cup \{0_P\}$$

and let $\text{Top } \psi$ be the restriction of ψ to $\text{Top } \mathcal{P}$. We also need the “bottom” of \mathcal{P} :

$$\text{Btm } \mathcal{P} = \{x \in P \mid \psi(x) = 0_Q\}.$$

Note that

$$\begin{aligned} \text{Top } \mathcal{P} \cup \text{Btm } \mathcal{P} &= P, \\ \text{Top } \mathcal{P} \cap \text{Btm } \mathcal{P} &= \{0_P\}. \end{aligned}$$

Finally, for a bounded order P , let $\text{Lat } P$ be the lattice we construct in [6] as an extension of $\text{Frame } P$ (see Figure 1 with $X = \emptyset$) by inserting the lattice $G(p, q)$, see Figure 2, as a sublattice into $\text{Frame } P$ for all $p, q \in P$ satisfying $0_P < p < q < 1_P$.

Lemma 11. *Let $\mathcal{P} = (P, Q, \psi)$ be a surjective order-triple. If $\text{Top } \psi$ is an isomorphism between $\text{Top } \mathcal{P}$ and Q , then \mathcal{P} has a surjective representation $\mathcal{L} = (K, L, \varphi)$, where $K = \text{Lat } P$, $L = \text{Lat}_X Q$ with $X = \text{Btm } \mathcal{P}$.*

Proof. Let $\mathcal{P} = (P, Q, \psi)$ be a surjective order-triple and let $\text{Top } \psi$ be an isomorphism between $\text{Top } \mathcal{P}$ and Q .

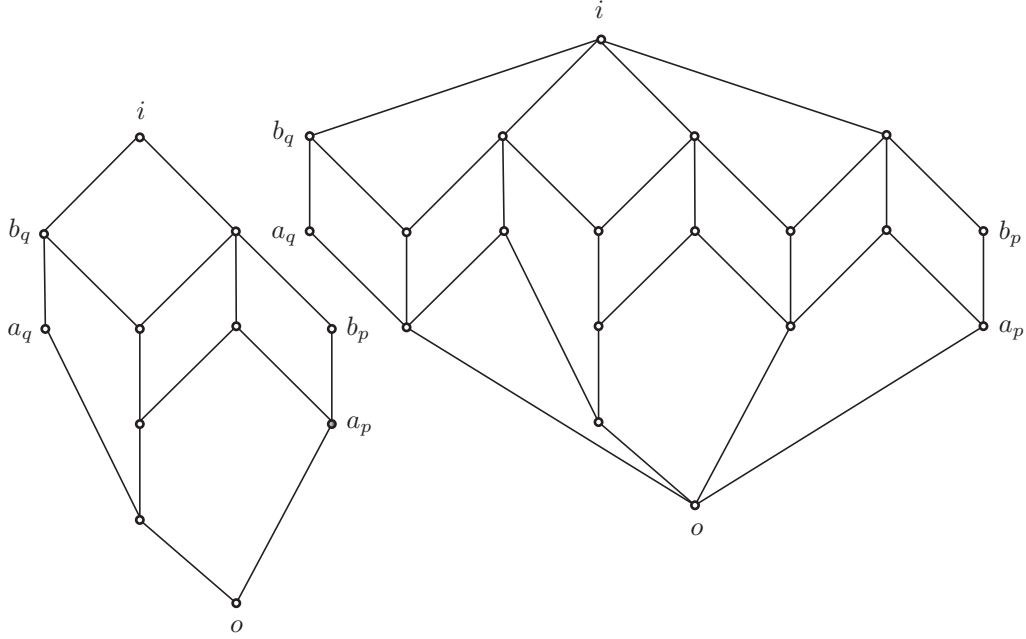
A $\{0, 1\}$ -isolating congruence α of a bounded lattice L , is a nontrivial congruence α , such that $\{0\}$ and $\{1\}$ are congruence blocks of α . With a $\{0, 1\}$ -isolating congruence α of $\text{Lat } P$, we associate a subset of the order $P^- = P - \{0, 1\}$:

$$\text{Base } \alpha = \{p \in P^- \mid a_p \equiv b_p \pmod{\alpha}\}.$$

We now restate some parts of Lemmas 6–9 of [6].

The correspondence $\delta: \alpha \rightarrow \text{Base } \alpha$ is an order preserving bijection between the order of $\{0, 1\}$ -isolating congruences of $\text{Lat } P$ and $\text{Down } P^-$, the order of down sets of P^- . Let $(\text{Down } P^-)^t$ be the order $\text{Down } P^-$ with a new unit element, P , added. We extend δ by $1 \rightarrow P$. Then φ is an isomorphism between $\text{Con}(\text{Lat } P)$ and $(\text{Down } P^-)^t$. The maps δ and δ^{-1} both preserve the property of being principal.

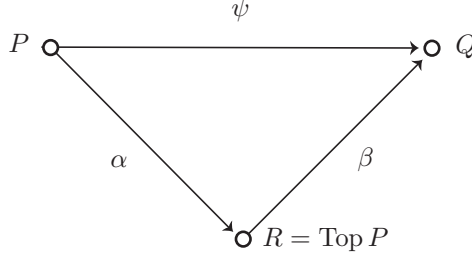
It follows that $P \cong \text{Princ}(\text{Lat } P)$.

FIGURE 2. The lattices $G(p, q)$ and $G(p, q)^{\text{Ext}}$

Now we describe a lattice-triple $\mathcal{L} = (K, L, \varphi)$. We define $K = \text{Lat } P$. Since $\text{Btm } \mathcal{P}$ is a down-set, it follows from the above statement that there is a congruence α of K with $\text{Btm } \mathcal{P} = \text{Base } \alpha$. Define the bounded lattice $L = K/\alpha$, and let φ be the natural $\{0, 1\}$ -homomorphism of K onto L . So $\mathcal{L} = (K, L, \varphi)$ is a lattice-triple. By the definitions of K , L , and φ , we have that $\mathcal{P}^3(\mathcal{L}) = \mathcal{P}$, in fact, the lattice-triple \mathcal{L} is a surjective representation of the order-triple \mathcal{P} .

An element u in a bounded lattice A is a *universal complement* if u is complementary to every other element of $A - \{0, 1\}$. Now consider the lattice $\text{Frame}_X P$, see Figure 1, which is the same as $\text{Frame } P$ except that we add the set X such that each $x \in X$ is a universal complement; $\text{Frame } P$ is the special case $X = \emptyset$. We then construct the lattice $\text{Lat}_X P$ the same way as we constructed $\text{Lat } P$ but starting with $\text{Frame}_X P$; the elements $x \in X$ remain universal complements in the larger lattice. Note that the lattice K can be represented in the form $\text{Lat}_X Q$, where $X = \text{Btm } \mathcal{P}$, concluding the proof of the lemma. \square

There are a several papers with lattice constructs based on $\text{Frame } P$ (see my paper [6] and G. Czédli's papers [1]–[3]—with more to come). We have just discussed the construction in [6], concluding that adding a set X of universal complements to $\text{Frame } P$ allows the same lattice construction and the order of principal congruences in the lattice constructed remains the same.

FIGURE 3. The order-triples $\mathcal{P}_\alpha = (P, R, \alpha)$ and $\mathcal{P}_\beta = (R, Q, \beta)$

4. PROVING THE REPRESENTATION THEOREM

Let $\mathcal{P} = (P, Q, \psi)$ be an order-triple. We consider the bounded order $R = \text{Top } \mathcal{P}$ and define the isotone map $\alpha: P \rightarrow R$ as follows:

$$(14) \quad \alpha(x) = \begin{cases} x, & \text{for } x \in \text{Top } \mathcal{P}; \\ 0_P = 0_R, & \text{otherwise.} \end{cases}$$

We also define the bounded isotone map $\beta: R \rightarrow Q$ as the restriction of ψ to $R = \text{Top } \mathcal{P}$, see Figure 3. Note that $\beta\alpha = \psi$.

This defines the order-triples $\mathcal{P}_\alpha = (P, R, \alpha)$ and $\mathcal{P}_\beta = (R, Q, \beta)$.

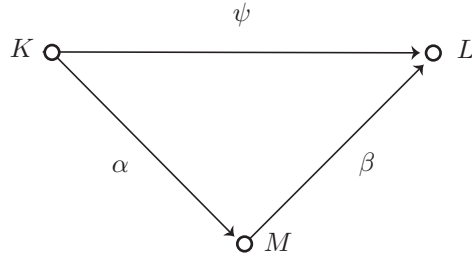
Since \mathcal{P}_α is a surjective order-triple and α is an isomorphism between $\text{Top } \mathcal{P}$ and R , it follows from Lemma 11 that \mathcal{P}_α has a surjective representation $\mathcal{L}_\alpha = (K, M, \varphi_\alpha)$ with $K = \text{Lat } P$ and $M = \text{Lat}_X(\text{Top } \mathcal{P})$, where $X = \text{Btm } \mathcal{P}$.

Now we make a small but important technical change. In the construction of $\text{Lat } P$ we replace $G(p, q)$ with the extended version $G(p, q)^{\text{Ext}}$ of G . Czédli [1], obtaining the lattice $\text{Lat}^{\text{Ext}} P$, $\text{Lat } P$ extended. Similarly in M , obtaining M^{Ext} .

Now we apply Czédli's Theorem 2. In \mathcal{P}_β , the map β is 0-separating, therefore, we can apply Theorem 2 to R , Q , and β . So we have obtained

- (a) a lattice-triple, $\mathcal{L}_\alpha = (K, M, \alpha)$ satisfying $\mathcal{P}^3(\mathcal{L}_\alpha) = \mathcal{P}_\alpha$;
- (b) a lattice-triple $(M, L, \varphi_{\text{Sub}})$ satisfying $\mathcal{P}^3(\mathcal{L}_\beta) = \mathcal{P}_\beta$.

We conclude that $\mathcal{L} = (K, L, \beta\alpha)$ is a lattice-triple, and $\mathcal{P}^3(\mathcal{L}) = \mathcal{P}$, verifying the Representation Theorem.

FIGURE 4. The lattice-triples (K, M, α) and (M, L, β)

5. PROBLEMS

Theorem 1 was extended in G. Czédli [1] to countable lattices and countable orders with zero.

Problem 1. Can we extended Theorem 5 to the countable case?

This would seem to be technically rather difficult.

Problem 2. For a finite semimodular lattice L , characterize $\text{Princ } L$.

Problem 3. For a finite planar semimodular lattice L , characterize $\text{Princ } L$.

APPENDIX A. A SMALLER CONSTRUCTION

Czédli's proof of Theorem 2 is easy to outline (but not so easy to compute). Let $\mathcal{P} = (P, Q, \psi)$ be an order triple. We form the bounded order $R = P \cup Q$, a disjoint union with $0_P, 0_Q$ and $1_P, 1_Q$ identified. So R is a bounded order containing P and Q as $\{0, 1\}$ -suborders. Therefore, we have $\text{Frame } P$ as a $\{0, 1\}$ -suborder of $\text{Frame } R$.

For $p < q$ in P and for $p < q$ in R , we insert $G(p, q)^{\text{Ext}}$ into $\text{Frame } R$ so that $\text{con}(a_p, b_p) < \text{con}(a_q, b_q)$ will hold. Then we need to ensure that

$$(15) \quad \text{con}(a_p, b_p) = \text{con}(a_{\psi(p)}, b_{\psi(q)}).$$

We accomplish this by inserting $G(p, \psi(p))^{\text{Ext}}$ and $G(\psi(p), p)^{\text{Ext}}$. The first insertion gives us

$$\text{con}(a_p, b_p) \leq \text{con}(a_{\psi(p)}, b_{\psi(q)}),$$

while the second gives us

$$\text{con}(a_p, b_p) \geq \text{con}(a_{\psi(p)}, b_{\psi(q)}).$$

The two together yield (15).

This step requires that we insert 30 elements. Now we reduce this number to 4. Consider the lattice $\text{Equi}(p)$ of Figure 5. For all $p \in P$, we add the four new elements of $\text{Equi}(p)$ to the construction to obtain an 8 element sublattice. This forces that $\text{con}(a_p, b_p) = \text{con}(a_{\psi(p)}, b_{\psi(q)})$. We do not present the verification of this construction.

By also replacing the sublattices $G(p, q)^{\text{Ext}}$ by $G(p, q)$, if P^- has n_P elements and c_P (nonreflexive) compatibilities while Q^- has c_Q (nonreflexive) compatibilities,

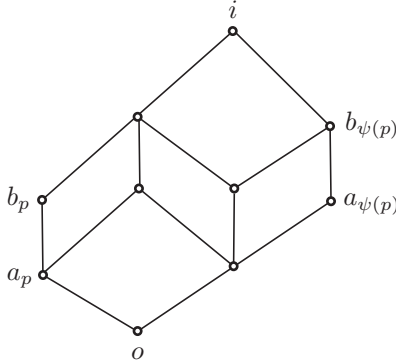


FIGURE 5. Adding 4 elements, $\text{Equi}(p)$

then Czédli's construction adds $15c_P + 15c_Q + 30n_P$ elements. The construction in this section adds $7c_P + 7c_Q + 4n_P$ elements.

Note that in Czédli's constructions, in general, it is important to use $G(p, q)^{\text{Ext}}$ rather than $G(p, q)$.

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